

J. Miritzis

Oscillatory behaviour of closed isotropic models in second order gravity theory

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Abstract Homogeneous and isotropic models are studied in the Jordan frame of the second order gravity theory. The late time evolution of the models is analysed with the methods of the dynamical systems. The normal form of the dynamical system has periodic solutions for a large set of initial conditions. This implies that an initially expanding closed isotropic universe may exhibit oscillatory behaviour.

Keywords Isotropic cosmologies Higher-order gravity Dynamical systems

1 Introduction

Quadratic gravitational Lagrangians were proposed shortly after the formulation of general relativity (GR) as alternatives to Einstein's theory. Gravity modifications in the form of higher-order curvature invariants in the Lagrangian are generally known as higher-order gravity (HOG) theories. They arise in string-theoretic considerations, e.g., brane models with Gauss-Bonnet terms [1] or models with a scalar field coupled to the Gauss-Bonnet invariant [2] (see [3] for a review) and generally involve linear combinations of all possible second order invariants that can be formed from the Riemann, Ricci and scalar curvatures. A quarter of a century ago there was a resurgence of interest in such theories in an effort to explain inflation. The reasons for considering HOG theories were multiple. Firstly, it was hoped that higher order Lagrangians would create a first approximation to quantum gravity, due to their better renormalisation properties than GR [4]. Secondly, it was reasonable to expect that on approach to a spacetime singularity, curvature invariants of all orders ought to play an important dynamical role. Far from

J. Miritzis
Department of Marine Sciences, University of the Aegean, University Hill, Mytilene
81100, Greece
E-mail: imyr@aegean.gr

the singularity, when higher order corrections become negligible, one should recover GR. Furthermore, it was hoped that these generalized theories of gravity might exhibit better behavior near singularities. Thirdly, inflation emerges in these theories in a most direct way. In one of the first inflationary models, proposed in 1980 by Starobinsky [5], inflation is due to the R^2 correction term in a gravitational Lagrangian $L = R + \beta R^2$ where β is a constant.

Recently there is a revival of interest in HOG theories in an effort to explain the accelerating expansion of the Universe [6, 7]. The general idea is to add an $1/R$ term to the Einstein-Hilbert Lagrangian or more generally to consider $R + \alpha R^{-n}$ Lagrangians [8, 9]. As the Universe expands, one expects that the inverse curvature terms will dominate and produce the late time accelerating expansion. Most studies are restricted to simple Friedmann-Robertson-Walker (FRW) models because of the complexity of the field equations. At present, the observational viability of these models is a subject of active research (see [10, 11, 12, 13] and references therein). However, it seems that a large class of Lagrangians may fit the observational data, but simple models based on the FRW metric are insufficient to pick the correct Lagrangian (see for example [14] for the reconstruction of the $f(R)$ theory which best reproduces the observed cosmological data). For more general spacetimes it may even be meaningless to say that R is small in some epoch of cosmic evolution and large in some other one (for a thorough critic see [15]).

In this paper we investigate the late time evolution of flat and positively curved FRW models with a perfect fluid in the $R + \beta R^2$ theory. This is the simplest generalization of the Einstein-Hilbert Lagrangian and the addition of the quadratic term represents a correction to general relativity. The simple vacuum case was studied in [16], where oscillatory behaviour of the solutions of closed models was found. Since HOG theories in vacuum are conformally equivalent to GR with a scalar field, it is tempting to say that the R^2 contribution has predictable cosmological consequences [17]. However, this is an oversimplification of the picture (see [10, 18] for specific examples). The two frames are mathematically equivalent, but physically they provide different theories. In the Jordan frame, gravity is described entirely by the metric $g_{\mu\nu}$. In the Einstein frame, the scalar field exhibits a non-metric aspect of the gravitational interaction, reflecting the additional degree of freedom due to the higher order of the field equations in the Jordan frame. Inclusion of additional matter fields, further complicates the situation and while the field equations in the Einstein frame are formally the Einstein equations, nevertheless this theory is not physically equivalent to GR. There is no universally acceptable answer to the issue “which conformal frame is physical” [19] (see [20] for a thorough analysis of different views).

The plan of the paper is as follows. Next Section contains a short comment on the stability of well-known power-law solutions. The field equations are written as a constrained four-dimensional polynomial dynamical system. Section 3 contains the analysis of the flat case. The so-called normal form of the dynamical system greatly simplifies the problem, since two of the equations decouple. In Section 4 we study the qualitative behaviour of the solutions near the equilibrium points of positively curved models and analyse

their late time evolution. It is shown that an initially expanding closed FRW universe may exhibit oscillatory behaviour.

2 Field equations

The general gravitational Lagrangian in four-dimensional spacetimes contains curvature invariants of all orders, $R, R^2, R_{\mu\nu}R^{\mu\nu}, \dots$. The term $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is omitted because of the Gauss-Bonnet theorem. A further simplification can be done in homogeneous and isotropic spacetimes where the variation of $R_{\mu\nu}R^{\mu\nu}$ with respect to the metric is proportional to the variation of R^2 [21]. We conclude that for isotropic cosmologies the gravitational Lagrangian contains only powers of the scalar curvature and we may consider HOG theories derived from Lagrangians of the form

$$L = f(R) \sqrt{-g} + L_{\text{matter}},$$

where f is an arbitrary smooth function. It is well-known that the corresponding field equations are fourth-order and take the form

$$f'(R) R_{\mu\nu} - \frac{1}{2}f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = T_{\mu\nu}, \quad (1)$$

where $\square = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ and a prime (') denotes differentiation with respect to R . The generalised Bianchi identities imply that $\nabla^\mu T_{\mu\nu} = 0$. Contraction of (1) yields the trace equation

$$3\square f'(R) + f'(R) R - 2f(R) = T. \quad (2)$$

In contrast to GR where the relation of R to T is algebraic, in HOG theories the trace equation (2) is a differential equation for R , with T as source term [22]. This suggests that in HOG theories both the metric and the scalar curvature are dynamical fields.

In the following, we consider a quadratic Lagrangian without cosmological constant, i.e. $f(R) = R + \beta R^2$, $\beta > 0$, and confine our attention to cosmologies with a perfect fluid with energy density ρ and pressure p , of the form

$$p = (\gamma - 1)\rho, \quad 0 \leq \gamma \leq 2.$$

For homogeneous and isotropic spacetimes¹ described by the standard FRW metric we need the following useful relations ($i, j = 1, 2, 3$ and $k = 0, \pm 1$)

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} \right) g_{ij}, \quad R = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right). \quad (3)$$

The 00 component of (1) is

$$H^2 + \frac{k}{a^2} + 2\beta \left[R \left(H^2 + \frac{k}{a^2} \right) + H\dot{R} - \frac{R^2}{12} \right] = \frac{1}{3}\rho, \quad (4)$$

¹ We adopt the metric and curvature conventions of [23]. Here, $a(t)$ is the scale factor, an overdot denotes differentiation with respect to time t , and units have been chosen so that $c = 1 = 8\pi G$.

where $H = \dot{a}/a$, is the Hubble function.

At this point we make a digression. For flat, $k = 0$, models, differentiating the relation $R = 6\dot{H} + 12H^2$ (which comes from the third of (3)) with respect to t , equation (4) takes the form

$$H^2 + 6\beta \left(2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 \right) = \frac{1}{3}\rho. \quad (5)$$

For radiation, $\gamma = 4/3$, there exists a special solution $a(t) = t^{1/2}$ as $t \rightarrow 0$. As Barrow and Middleton point out [24], this is also an exact vacuum solution of the purely quadratic theory, in the sense that it solves $2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 = 0$. However, this solution is unstable as we shall see in a moment (see [25] for a detailed stability analysis of isotropic models in general $f(R)$ theories). Following Carroll et al [8], we reduce the order of (5) in vacuum by defining

$$X = -H, \quad Y = \dot{H} \Rightarrow \ddot{H} = -Y \frac{dY}{dX}.$$

Then, the asymptotic values of the function $U(X) = -X^2/Y$ as $X \rightarrow 0$, correspond to the exponents p for power-law solutions $a(t) = t^p$. We apply this technique to (5) and we find the first-order equation

$$\frac{dU}{dX} = \frac{1}{12\beta} \frac{U^3}{X^3} - 3 \frac{U}{X} \left(U - \frac{1}{2} \right), \quad (6)$$

with the corresponding direction field shown in Fig. 1. We see that the solution $U(X) = 1/2$ is a past attractor ($X \rightarrow -\infty$), but becomes unstable as $X \rightarrow 0$, in agreement with [25], and that $|U(X)| \rightarrow 0$, corresponding to the singularity $|\dot{H}| \rightarrow \infty$. This is also evident by studying the asymptotic behaviour of solutions of the linearised equation near the constant solutions $1/2$ and 0 .

We now continue our discussion about the choice of variables. Setting $x = 1/a$, the 00 equation becomes

$$H^2 + kx^2 + 2\beta \left[R(H^2 + kx^2) + H\dot{R} - \frac{R^2}{12} \right] = \frac{1}{3}\rho, \quad (7)$$

and the evolution equation for x is

$$\dot{x} = -xH. \quad (8)$$

The evolution equation for H comes from the third of (3) and takes the form

$$\dot{H} = \frac{1}{6}R - 2H^2 - kx^2. \quad (9)$$

The conservation equation

$$\dot{\rho} = -3\gamma\rho H, \quad (10)$$

is a consequence of the Bianchi identities. With the relation $\square R = -\ddot{R} - 3H\dot{R}$, equation (2) becomes

$$\ddot{R} + 3H\dot{R} + \frac{1}{6\beta}R = \frac{1}{6\beta}(4 - 3\gamma)\rho. \quad (11)$$

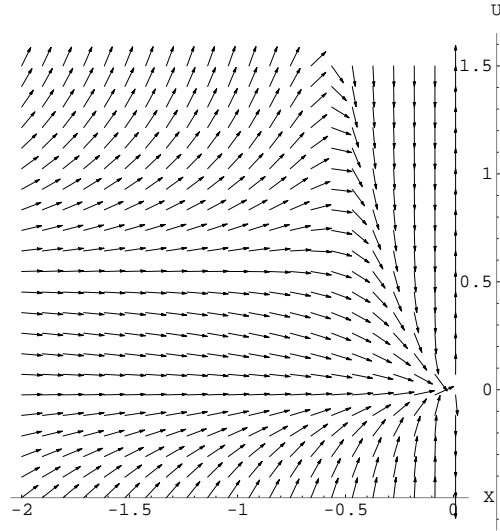


Fig. 1 Direction field of the reduced equation (6)

This equation is usually considered as superfluous, since it follows from the differentiation of the equation (7) with respect to t ; for example equation (11) was used in [26] as a control of the accuracy of numerical investigations of Bianchi type I and IX models. Thus, one can choose R, x, H and ρ as dynamical variables which obey the evolution equations (7), (8), (9) and (10) and constitute a four-dimensional dynamical system.

The form of equation (7) suggests the choice of expansion normalized variables of the type

$$u \sim R/H, \quad \Omega \sim \rho/H^2, \dots \quad (12)$$

Detailed studies using this approach for general $f(R)$ theories can be found in [12, 27]. However, even in the simplest case of flat models in vacuum, numerical investigation of the system with initial values $H(0) > 0$ shows that $H(t)$ exhibit damped oscillations, with almost zero minima. Therefore although permissible, the transformation (12) may induce fake singularities to the solutions. Moreover, as mentioned in [23], a drawback of this choice of variables is that it does not give a complete description of the evolution for bouncing or recollapsing models. If the Hubble parameter passes through zero the logarithmic time coordinate is ill-defined and the transformation (12) is singular (see [28] for a comparison of compact and non-compact variables).

In order to circumvent these difficulties, we introduce one more degree of freedom by using (11), thus augmenting the dimension of the dynamical system. The state (R, \dot{R}, x, ρ, H) of the system lies on the hypersurface of \mathbb{R}^5 defined by the constraint (7). The presence of (R, \dot{R}) in the state vector reflects the fact that there are additional degrees of freedom in HOG theories than in GR (cf. the remark after equation (2)).

We define $x_1 = R$, $x_2 = \dot{R}$ and use (7) to eliminate ρ , so that equations (11), (8) and (9) constitute a four-dimensional system. The parameter β may be used to define dimensionless variables by rescaling

$$x_1 \rightarrow \frac{2}{\beta} x_1, \quad x_2 \rightarrow \sqrt{\frac{2}{3\beta^3}} x_2, \quad H \rightarrow \frac{H}{\sqrt{6\beta}}, \quad x \rightarrow \frac{x}{\sqrt{6\beta}}, \quad t \rightarrow \sqrt{6\beta} t,$$

and our system becomes

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + (1 - 3\gamma) x_2 H + \frac{(4 - 3\gamma)}{4} Z, \\ \dot{x} &= -xH, \\ \dot{H} &= 2x_1 - 2H^2 - kx^2, \end{aligned} \tag{13}$$

with $Z = [H^2 + kx^2 - 4x_1^2 + 4x_1(H^2 + kx^2)]$.

Remark. The system (13) is not an arbitrary “free” four-dimensional system. In view of (7) the initial conditions have to satisfy the condition

$$H^2 + kx^2 + 4x_1(H^2 + kx^2) + 4Hx_2 - 4x_1^2 \geq 0. \tag{14}$$

With a little manipulation of the equations (13) it can be shown that, once we start with initial conditions satisfying (14) at time t_0 , the solutions of the system satisfy this inequality for all $t > t_0$. Thus the field equations share the general property of the Einstein equations, namely that the subsequent evolution of the system is such that the solutions respect the constraint.

3 Flat models

In the flat, ($k = 0$), case the dimension of the dynamical system is reduced by one, since the evolution equation for x decouples from the remaining equations. The corresponding system is

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + (1 - 3\gamma) x_2 H - (4 - 3\gamma) x_1^2 + \frac{(4 - 3\gamma)}{4} H^2 + (4 - 3\gamma) x_1 H^2, \\ \dot{H} &= 2x_1 - 2H^2, \end{aligned} \tag{15}$$

i.e., the vector field does not depend on the dynamical variable x . Vacuum models are significantly simpler to analyse, but we do not study them separately as they arise formally by setting $\gamma = 4/3$ in all the equations, while (14) describing the phase space becomes equality. The only equilibrium point of (15) is the origin and corresponds to flat empty spacetime. The eigenvalues of the Jacobian matrix at the origin are $\pm i, 0$ and therefore we cannot infer about stability using the linearisation theorem. For nonhyperbolic equilibrium points there exist no general methods for studying their stability. The normal form of the system may provide some information about the behaviour of the solutions near the equilibrium. The normal form theory consists in a nonlinear coordinate transformation that allows to simplify the

nonlinear part of the system (cf. [29] for a brief introduction). This task will be accomplished in three steps in some detail for the convenience of readers with no previous knowledge of the method.

1. Let P be the matrix formed from the eigenvectors which transforms the linear part of the vector field into Jordan canonical form. We write (15) in vector notation (with $\mathbf{x} = (x_1, x_2, H)$) as

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{F}(\mathbf{x}), \quad (16)$$

where A is the linear part of the vector field and $\mathbf{F}(\mathbf{0}) = \mathbf{0}$.

2. Using the matrix P , we define new variables, $(y_1, y_2, y) \equiv \mathbf{y}$, by the equations

$$y_1 = x_1, \quad y_2 = -x_2, \quad y = H + 2x_2,$$

or in vector notation $\mathbf{y} = P\mathbf{x}$, so that (16) becomes

$$\dot{\mathbf{y}} = P^{-1}AP\mathbf{y} + P^{-1}\mathbf{F}(P\mathbf{y}).$$

Denoting the canonical form of A by B we finally obtain the system

$$\dot{\mathbf{y}} = B\mathbf{y} + \mathbf{f}(\mathbf{y}), \quad (17)$$

where $\mathbf{f}(\mathbf{y}) := P^{-1}\mathbf{F}(P\mathbf{y})$. In components system (17) is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ (4-3\gamma)y_1^2 - (3\gamma+2)y_2^2 - \frac{4-3\gamma}{4}y^2 - 3yy_2 - (4-3\gamma)y_1(2y_2+y)^2 \\ -2(4-3\gamma)y_1^2 + 2(3\gamma-2)y_2^2 - \frac{3\gamma}{2}y^2 - 2yy_2 + 2(4-3\gamma)y_1(2y_2+y)^2 \end{bmatrix}.$$

Inequality (14) imposes the constraint

$$y^2 - 4(y_1^2 + y_2^2) + 4y_1(y + 2y_2)^2 \geq 0. \quad (18)$$

3. Under the non-linear change of variables

$$\begin{aligned} y_1 &\rightarrow y_1 + 3\gamma y_1^2 + (3\gamma - 2)y_2^2 + \frac{4-3\gamma}{4}y^2 + \frac{3}{4}y_2y, \\ y_2 &\rightarrow y_2 + 4y_1y_2 + \frac{3}{4}y_1y, \\ y &\rightarrow y - 2y_1y_2 + 2y_1y, \end{aligned} \quad (19)$$

and keeping only terms up to second order, the system transforms to

$$\begin{aligned} \dot{y}_1 &= -y_2 - \frac{3}{2}y_1y + \mathcal{O}(3), \\ \dot{y}_2 &= y_1 - \frac{3}{2}y_2y + \mathcal{O}(3), \\ \dot{y} &= 6(\gamma - 1)(y_1^2 + y_2^2) - \frac{3\gamma}{2}y^2 + \mathcal{O}(3). \end{aligned}$$

Finally, defining cylindrical coordinates $(y_1 = r \cos \theta, y_2 = r \sin \theta, y = y)$, we obtain

$$\begin{aligned}\dot{r} &= -\frac{3}{2}ry + \mathcal{O}(3), \\ \dot{\theta} &= 1 + \mathcal{O}(2), \\ \dot{y} &= 6(\gamma - 1)r^2 - \frac{3\gamma}{2}y^2 + \mathcal{O}(3).\end{aligned}\tag{20}$$

We may continue to simplify the third order terms and the result should be

$$\begin{aligned}\dot{r} &= a_1ry + a_2r^3 + a_3ry^2 + \mathcal{O}(4), \\ \dot{\theta} &= 1 + \mathcal{O}(2), \\ \dot{y} &= b_1r^2 + b_2y^2 + b_3r^2y + b_4y^3 + \mathcal{O}(4).\end{aligned}$$

This is the normal form in cylindrical coordinates of every three-dimensional vector field with linear part

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y \end{bmatrix},$$

(see [30] p. 377). However, since we are interested on the behaviour of the solutions only near the origin, we truncate the vector field at $\mathcal{O}(2)$. We note that the θ dependence of the vector field has been eliminated, so that we can study the system in the (r, y) space. The second equation of (20) implies that the trajectory in the $y_1 - y_2$ plane spirals with angular velocity 1. The projection of (20) on the $r - y$ plane is

$$\begin{aligned}\dot{r} &= -\frac{3}{2}ry, \\ \dot{y} &= 6(\gamma - 1)r^2 - \frac{3\gamma}{2}y^2.\end{aligned}\tag{21}$$

This system belongs to a family of systems studied in 1974 by Takens [31] (see also [30] for a description of all phase portraits for different values of the parameters).

System (21) is invariant under the transformation $t \rightarrow -t, y \rightarrow -y$ (which implies that all trajectories are symmetric with respect to the r axis) and the line $r = 0$ is invariant. Note also that the system (21) has invariant lines $y = \pm 2r$ (compare with the similar system in [32]). The behaviour of the solutions depends on the parameter γ and as we shall see, $\gamma = 1$ is a bifurcation value.

Case I, $\gamma < 1$. We observe that y is always decreasing along the orbits while r is decreasing in the first quadrant. Since no trajectory can cross the line $y = 2r$, all trajectories starting above this line, approach the origin asymptotically. The phase space of the dynamical system (21) is not the whole $r - y$ plane, because of the constraint (18). In terms of the variables (19) and neglecting fourth-order terms the constraint becomes

$$(y^2 - 4r^2)(1 + 6\gamma y_1) \geq 0.\tag{22}$$

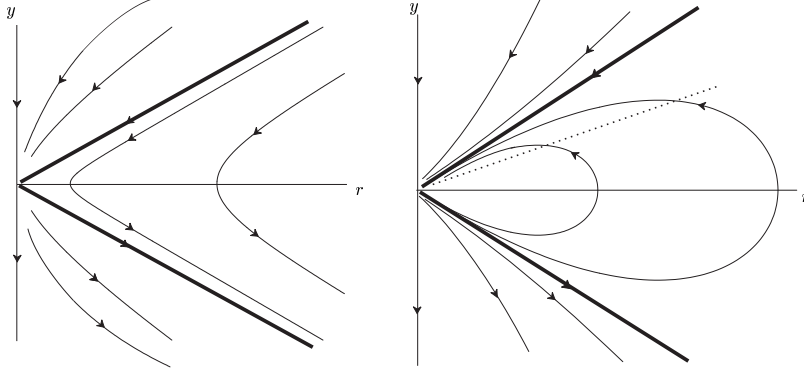


Fig. 2 Phase portrait of (21) for $\gamma < 1$ and $\gamma > 1$. The invariant lines $y = \pm 2r$ are shown with heavy lines and $y = 2\sqrt{\gamma-1}r$ by the dotted line.

Since we are interested on trajectories starting close to the origin, for initial values of y_1 satisfying $|y_1(0)| \leq 1/6\gamma$ the first of (21) guarantees that $|y_1(t)|$ remains less than $1/6\gamma$, so that (22) implies

$$y^2 - 4r^2 \geq 0.$$

Therefore, we should consider only trajectories starting above the line $y = 2r$ and according to the previous discussion all these trajectories asymptotically approach the origin.

Case II, $\gamma > 1$. In the first quadrant r is decreasing along the orbits and, that \dot{y} vanishes along the line $y = 2\sqrt{\gamma-1}r$. Once a trajectory crosses the line $y = 2\sqrt{\gamma-1}r$, it is trapped between the lines $y = 2\sqrt{\gamma-1}r$ and $y = 2r$, and since $\dot{r} < 0$, it approaches the origin asymptotically.

Case III, $\gamma = 1$. It is evident that all trajectories are straight lines approaching asymptotically the origin.

We conclude that the late time behaviour of flat models is similar to the future predicted by GR. More precisely, all initially expanding flat models close to the state $R = \dot{R} = H = \rho = 0$, asymptotically approach the flat empty spacetime.

4 Positively curved models

In this section we consider an initially expanding closed universe described by the full four-dimensional system (13) with $k = 1$. There are two equilibria:

$\mathcal{M} : x_1 = 0, x_2 = 0, x = 0, H = 0$. This corresponds to the limiting state of an almost empty, slowly varying universe with $H \rightarrow 0$, while the scale factor goes to infinity. The point \mathcal{M} which resembles to the Minkowski solution, is located at the boundary of the phase space.

$\mathcal{S} : H = 0, x_2 = 0, x = \bar{x} \equiv \sqrt{\frac{2-3\gamma}{3\gamma-4}}, x_1 = \bar{x}_1 \equiv \bar{x}^2/2$, with $2/3 < \gamma < 4/3$. It is a static solution and the eigenvalues of the Jacobian matrix at \mathcal{S} are

$$\pm \frac{8 - 9\gamma \pm \sqrt{3\gamma(36\gamma^2 - 69\gamma + 32)}}{2(3\gamma - 4)}.$$

The real parts of the eigenvalues are nonzero for almost all² permissible values of γ , and we conclude that the local stable and unstable manifolds through \mathcal{S} are both two-dimensional. The point \mathcal{S} corresponds to the Einstein static universe, where the effective cosmological constant is provided by the curvature equilibrium \bar{x}_1 . To see this, it is sufficient to write equation (7) in the original variables at the equilibrium point as

$$k\bar{x}^2 + \frac{\Lambda}{3} = \frac{1}{3}\bar{\rho},$$

with $\Lambda > 0$. The cosmological constant depends on both the parameters β and γ , for example, for $\gamma = 1$, $\Lambda = (12\beta - 1)^2/2\beta$. Static solutions have little interest as cosmological models and we turn our attention to the other equilibrium, \mathcal{M} .

The point \mathcal{M} is a nonhyperbolic equilibrium and we find again the normal form of the system, which is given by (A.29) in the Appendix. Defining cylindrical coordinates $(y_1 = r \cos \theta, y_2 = r \sin \theta, x = x, y = y)$, we obtain

$$\begin{aligned} \dot{r} &= -\frac{3}{2}ry + \mathcal{O}(3), \\ \dot{\theta} &= 1 + \mathcal{O}(2), \\ \dot{x} &= -xy + \mathcal{O}(3), \\ \dot{y} &= 6(\gamma - 1)r^2 - \frac{3\gamma - 2}{2}x^2 - \frac{3\gamma}{2}y^2 + \mathcal{O}(3). \end{aligned} \tag{23}$$

We truncate the vector field at $\mathcal{O}(2)$ and we note again that the θ dependence of the vector field has been eliminated, so that we can study the system in the (r, x, y) space. We write the first and third of (23) as a differential equation

$$\frac{dr}{dx} = \frac{3}{2} \frac{r}{x},$$

which has the general solution

$$r = Ax^{3/2}, \quad A > 0. \tag{24}$$

² More precisely, the real parts of the eigenvalues are zero for

$$\frac{2}{3} < \gamma \leq \frac{23 - \sqrt{17}}{24} \approx 0.79,$$

and nonzero in the rest of the interval $(\frac{2}{3}, \frac{4}{3})$.

Substitution of (24) into the fourth equation of (23) yields the projection of the fourth-dimensional system on the $x - y$ plane, namely

$$\begin{aligned}\dot{x} &= -xy, \\ \dot{y} &= b(\gamma - 1)x^3 - \frac{3\gamma - 2}{2}x^2 - \frac{3\gamma}{2}y^2, \quad b > 0.\end{aligned}\quad (25)$$

Some general properties of the solutions of (25) follow by inspection. By standard arguments all trajectories are symmetric with respect to the x axis. Note that the new x defined by (A.28) remains non-negative for initial values $y_1(0)$ sufficiently small and, since the line $x = 0$ is invariant, any trajectory starting at the half plane $x > 0$ remains there for all $t > 0$. System (25) has two equilibrium points, the origin $(0, 0)$ and $(x_*, 0)$, where

$$x_* = \frac{3\gamma - 2}{2b(\gamma - 1)},$$

and therefore $x_* > 0$ for $\gamma < 2/3$ or $\gamma > 1$. The origin is again a nonhyperbolic equilibrium point. Computation of the Jacobian matrix at the equilibrium $(x_*, 0)$ shows that for the linearised system, this point is a saddle for $0 \leq \gamma < 2/3$, and a center for $1 < \gamma \leq 2$. For $\gamma > 1$, it is easy to see that x is decreasing in the first quadrant and y is decreasing along the orbits in the strip $0 < x < x_*$. On any orbit starting in the first quadrant with $x < x_*$, y becomes zero at some time and the trajectory crosses vertically the x -axis. Once the trajectory enters the second quadrant, x increases and y decreases. For $2/3 < \gamma \leq 1$, all trajectories starting in the first quadrant follow the same pattern.

System (25) has a first integral, *viz.*

$$\begin{aligned}\phi(x, y) &= -\frac{2b}{3}x^{3(1-\gamma)} + x^{2-3\gamma} + \frac{y^2}{x^{3\gamma}}, \quad \gamma \neq 1, \\ \phi(x, y) &= \frac{1}{x} + \frac{y^2}{x^3}, \quad \gamma = 1.\end{aligned}$$

This can be seen by writing (25) as

$$\frac{dy}{dx} = \frac{bx^3 - \frac{3\gamma-2}{2}x^2 - \frac{3\gamma}{2}y^2}{-yx}.$$

Setting $y^2 = z$, we obtain a linear differential equation for z which is easily integrable. The level curves of ϕ are the trajectories of the system.

We shall show for the system (25) that: (i) for every $\gamma \in (\frac{2}{3}, 2]$ there are no solutions asymptotically approaching the origin (ii) for $\gamma \in (\frac{2}{3}, 1]$ there are no periodic solutions and (iii) for $\gamma \in (1, 2]$ there exist periodic solutions and the basin of every periodic trajectory is the set

$$\left\{ (x, y) \in \mathbb{R}^2 : y^2 + x^2 - \frac{2b}{3}x^3 < 0, x > 0 \right\}.$$

Proof The proof mimics that found in [16] for the simple case $\gamma = 4/3$.

Let $\phi(x, y) = C$. We have

$$y^2 = x^2 \left(-1 + \frac{2b}{3}x + Cx^{3\gamma-2} \right), \quad (26)$$

which implies that the function

$$f(x) = -1 + \frac{2b}{3}x + Cx^{3\gamma-2}$$

must be non-negative. We consider two cases.

1. $C \geq 0$. Then f is strictly increasing for $x \geq 0$ and $f(0) = -1$, thus f has a unique root $x_1 > 0$ depending on C . It follows that for $C \geq 0$ any orbit starting in the first quadrant satisfies

$$x \geq x_1(C) > 0,$$

i.e., there are no solutions approaching the axis $x = 0$. These solutions are not closed since they intersect the x -axis only once at $x_1(C)$.

2. $C < 0$. If $2/3 < \gamma \leq 1$, then f has again a unique root hence the trajectories follow the same pattern as in case 1. If $\gamma > 1$, then f has two zeros, say $x_1(C) < x_2(C)$. This means that $0 < x_1(C) < x < x_2(C)$, i.e., x is bounded, and by (26), so is y . Thus, an orbit of (25) starting in the first quadrant crosses the x -axis at $x_1(C)$ and re-enters in the first quadrant crossing the x -axis at $x_2(C)$ i.e. it is a closed curve and represents a periodic solution. The curve corresponding to $C = 0$ separates the phase space into two disjoint regions I and II. In region II, ($C < 0$), every trajectory corresponds to a periodic solution and we conclude that the basin of every periodic trajectory is the set $y^2 + x^2 - \frac{2b}{3}x^3 < 0$.

Remark. The mere existence of closed orbits around the equilibrium point $(x_*, 0)$ could be inferred from the following theorem: If an equilibrium point \mathbf{p} is a center for the linearised system and all trajectories are symmetric with respect to the x axis, then \mathbf{p} is also a center for the nonlinear system (25) (cf. [29] Theorem 6, page 141). In the above proof we also determine the subset of the phase space which contains all periodic orbits.

Using all this information we may sketch the phase portrait of the system (Fig. 3).

For $\gamma < 2/3$ the homoclinic curves to the origin as well as the curves approaching the origin indicate that an initially expanding closed universe may avoid recollapse. This result is also valid in GR when matter fields violate the strong energy condition [33]. For $\gamma \in (\frac{2}{3}, 1]$ every solution curve becomes unbounded and we may interpret this as an indication that this universe recollapses.

The range $(1, 2]$ for γ is the more interesting because of the periodic orbits in region II. The phase portrait in Fig. 3 may lead to the conclusion that the periodic orbits are far from the origin. However, the position of the cycles in the phase space depends on the constants b in (25) and C in (26) and therefore, for suitable values of b and C , there exist periodic orbits arbitrary close to the origin. Note that the periodic solutions of (25) induce periodicity

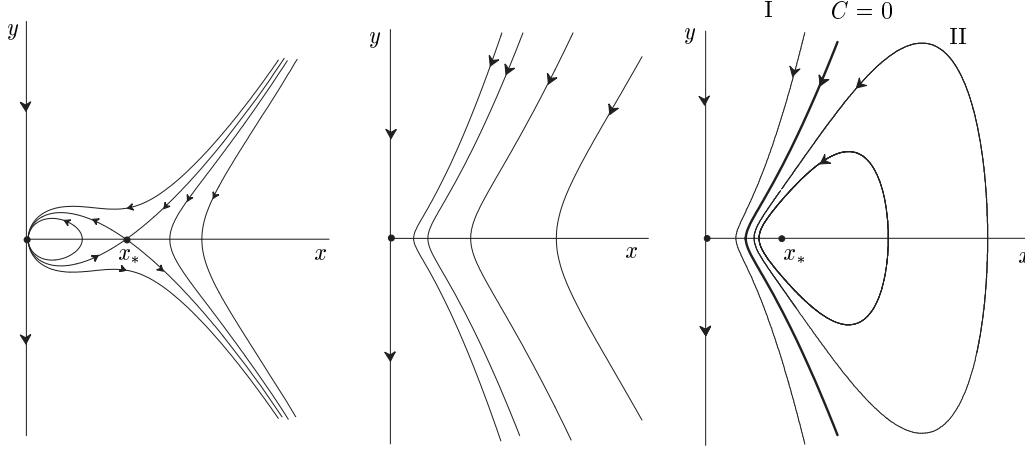


Fig. 3 Phase portrait of (25) for $\gamma < \frac{2}{3}$, $\frac{2}{3} < \gamma < 1$ and $\gamma > 1$.

to the full four-dimensional system (23) or (A.29). In fact, if $x(t)$ and $y(t)$ are periodic solutions, then (24) implies that the solutions

$$y_1(t) = r(t) \cos(t + \theta_0), \quad y_2(t) = r(t) \sin(t + \theta_0)$$

oscillate in the $y_1 - y_2$ plane with a periodic amplitude $r(t)$. Note also that the periodic motion in the $x - y$ plane is independent from the rotation of r in the $y_1 - y_2$ plane.

Obviously we cannot assign a physical meaning to the new variables (y_1, y_2, x, y) , since the transformations (A.27) and (A.28) have “mixed” the original variables of (13) in a nontrivial way. However, the periodic character of the solutions of (A.29) whatever the physical meaning of the variables be, has the following interpretation. *Close to the equilibrium of the original system (13), there exist periodic solutions.* This result is in agreement with [34] where it is shown that for Lagrangians $R + \beta R^m$, bounces of closed models are allowed for every integer value of m .

Remark. As mentioned in Section 2 numerical experiments show that the solutions of the system have oscillatory behavior. This property is intuitively evident by looking at the harmonic oscillator, equation (11). Using the rescalings of the dynamical variables along with the scaling $\rho \rightarrow \beta\rho$, equation (11) becomes

$$\ddot{R} + 3H\dot{R} + R = \frac{(4 - 3\gamma)}{2}\rho,$$

which is the equation of a forced, damped harmonic oscillator with unit angular frequency. Qualitative arguments supported by numerical solutions indicate that oscillatory motion, possibly slightly damped, is essentially independent of k and γ and becomes the late time behaviour. This is revealed in the normal forms of the systems, (20) and (23), where the θ equation shows oscillatory motion with unit angular frequency. We emphasize again that this motion is independent from the periodic motion in the $x - y$ plane.

The normal form analysis reveals both kinds of oscillatory behaviour and distinguishes the models which actually exhibit undamped oscillations at late time.

5 Discussion

We analysed the qualitative behaviour of flat and positively curved FRW models filled with ordinary matter described by a perfect fluid in the Jordan frame of the $R + \beta R^2$ theory. We have shown that initially expanding flat models close to the equilibrium $H = 0, R = 0, \rho = 0$ are ever expanding and asymptotically approach flat empty spacetime. Therefore, the late time behaviour of flat models is a common property of quadratic gravity and GR. Closed models can avoid recollapse for $\gamma < 2/3$, but not for γ in the range $[\frac{2}{3}, 1]$, thereby also behaving similarly to the corresponding general-relativistic cosmologies.

The interesting feature is the existence of periodic solutions near the origin for $\gamma > 1$. This is not revealed in the Einstein frame (see [32]), possibly because in that investigation, the scalar field related to the conformal transformation is not coupled to matter, i.e., the matter Lagrangian is added after performing the conformal transformation. Were the two frames physically equivalent then, a naive physical explanation of the cyclic behavior could rely on the scalar field which behaves like a “cosmological constant” in the high curvature limit. The perfect fluid which is also present dominates in the low curvature regime allowing for a recollapse, but then the effective cosmological constant induces a bounce in the high-curvature regime.

The periodic solutions imply that an initially expanding closed universe can avoid recollapse through an infinite sequence of successive expansions and contractions. The oscillatory open model proposed by Steinhardt and Turok [35] has renewed interest in cyclic universes. However, observations do not exclude Ω_{total} to be slightly larger than one [36]. Oscillatory closed models were considered in the context of Loop Quantum Cosmology [37] and oscillatory (but not periodic) solutions in GR were found in [38] for closed models containing radiation and dust or scalar field. In Fig. 3 the basin of all periodic trajectories of the (25) is the domain on the right of the $C = 0$ curve and since it is an open subset of the phase space, we conclude that there is enough room in the set of initial data of (13) which lead to an oscillating scale factor. The $R + \beta R^2$ theory has offered a successful inflationary model, but whether it is capable to explain the acceleration of the universe is an open question that needs to be studied in more detail. In particular, the observed slow acceleration must be related to the periods of the closed curves of (25).

The normal form theory is a powerful method for determining the qualitative behaviour of a dynamical system near a nonhyperbolic equilibrium point, but does not give any information about the structure of the solutions far from this equilibrium. Our results are based on an analysis of the behaviour of the dynamical system (13) only near the equilibrium solutions. The geometry of the trajectories of a four-dimensional dynamical system may be quite complicated, e.g. strange attractors may be present. For the system (13) the whole picture may come in view only from the investigation of the

global structure of the solutions. The study of this question is an interesting challenge for mathematical relativity.

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Appendix: Normal form of (13)

Following the usual algorithm, we use the matrix which transforms the linear part of (13) into Jordan canonical form and define new variables by

$$y_1 = x_1, \quad y_2 = -x_2, \quad x = x, \quad y = H + 2x_2, \quad (\text{A.27})$$

so that (13) becomes

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ (4-3\gamma)y_1^2 - (3\gamma+2)y_2^2 - \frac{4-3\gamma}{4}(kx^2 + y^2) - 3yy_2 - (4-3\gamma)Z_3 \\ -2xy_2 - xy \\ -2(4-3\gamma)y_1^2 + 2(3\gamma-2)y_2^2 - \frac{3\gamma-2}{2}kx^2 - \frac{3\gamma}{2}y^2 - 2yy_2 + 2(4-3\gamma)Z_3 \end{bmatrix},$$

with $Z_3 = y_1 [kx^2 + (2y_2 + y)^2]$. Under the non-linear change of variables

$$\begin{aligned} y_1 &\rightarrow y_1 + 3\gamma y_1^2 + (3\gamma-2)y_2^2 + \frac{4-3\gamma}{4}(x^2 + y^2) + \frac{3}{4}y_2y, \\ y_2 &\rightarrow y_2 + 4y_1y_2 + \frac{3}{4}y_1y, \\ x &\rightarrow x + 2y_1x, \\ y &\rightarrow y - 2y_1y_2 + 2y_1y, \end{aligned} \quad (\text{A.28})$$

and keeping only terms up to second order, the system transforms to

$$\begin{aligned} \dot{y}_1 &= -y_2 - \frac{3}{2}y_1y + \mathcal{O}(3), \\ \dot{y}_2 &= y_1 - \frac{3}{2}y_2y + \mathcal{O}(3), \\ \dot{x} &= -xy + \mathcal{O}(3), \\ \dot{y} &= 6(\gamma-1)(y_1^2 + y_2^2) - \frac{3\gamma-2}{2}x^2 - \frac{3\gamma}{2}y^2 + \mathcal{O}(3). \end{aligned} \quad (\text{A.29})$$

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